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Linear Algebra and its Applications 413 (2006) 458–473
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

A polynomial matrix approach to the structural properties of 2D positive systems

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Received 21 December 2004; accepted 15 November 2005
Available online 10 January 2006
Submitted by G. de Oliveira

Abstract

Reachability and observability of two-dimensional (2D) discrete state-space models are introduced in two different forms: a local form, which refers to single local states, and a global form, which pertains to the infinite set of local states lying on a separation set [M. Bisiacco, State and output feedback stabilizability of 2D systems, IEEE Trans. Circ. Syst., CAS-32 (1985) 1246–1249; E. Fornasini, G. Marchesini, Global properties and duality in 2-D systems, Syst. Control Lett. 2 (1) (1982) 30–38]. While local reachability and observability can be naturally characterized by resorting to classical state space techniques, their global counterparts are better addressed by means of polynomial techniques. In this paper, reachability and observability are introduced in the context of 2D positive systems and their global versions investigated via a polynomial approach. Necessary and sufficient conditions for the existence of these properties are provided and, in particular, polynomial canonical forms for globally reachable/observable positive systems with scalar inputs/scalar outputs are provided.
Published by Elsevier Inc.

Keywords: Positive 2D systems; Local reachability; Global reachability; Local observability; Global observability; Polynomial methods

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0024-3795/$ - see front matter Published by Elsevier Inc.
doi:10.1016/j.laa.2005.11.006

1. Introduction

“2D positive systems”, i.e. two-dimensional state-space models whose input, state and output variables take nonnegative values, have been introduced in the nineties. The aim was that of providing a unifying theoretical framework to a family of interesting problems. Specifically, research interests in this field have been stimulated by a series of contributions dealing with river pollution modeling [4], modeling of a single-carriageway traffic flow [9], gas absorption and water stream heating [14], diffusion of a tracer into a blood vessel [16], etc. These contributions share two common features: on the one hand all “internal” variables are intrinsically nonnegative, as they represent concentrations, pressures, numbers of vehicles, etc., on the other hand, the dynamics is well described by a (quarter plane causal) 2D state-space model, as the system variables depend on a time and a space coordinate and obey a quarter plane causality law.

Research efforts were first oriented to extend positive matrix theory to pairs of matrices (see e.g. [7,8]), thus leading to the analysis of the free state evolution and the asymptotic stability of 2D positive systems. More recently, research efforts in 2D positive systems have concentrated on the analysis of their structural properties, and some results about reachability and controllability have been presented in [10–12], by assuming a traditional state-space, and hence geometric, approach.

In the 2D setting, the concepts of reachability and observability are naturally introduced in two different forms: a weak (local) form, which refers to single “local states”, and a strong (global) form, which pertains to the infinite set of local states lying on a “separation set” [1,5]. In this paper, the aforementioned concepts are introduced and investigated in the context of 2D positive systems, driven by nonnegative inputs and described by the following state-updating equation [5]:

$$\begin{aligned}\mathbf{x}(h+1,k+1) &= A_1\mathbf{x}(h,k+1) + A_2\mathbf{x}(h+1,k)\\ &\quad + B_1\mathbf{u}(h,k+1) + B_2\mathbf{u}(h+1,k), \end{aligned} \tag{1.1}$$

$$\mathbf{y}(h,k) = C\mathbf{x}(h,k), \quad h,k \in \mathbb{Z}, h+k \geqslant 0, \tag{1.2}$$

where the n-dimensional *local states* $\mathbf{x}(\cdot,\cdot)$, the m-dimensional inputs $\mathbf{u}(\cdot,\cdot)$ and the p-dimensional outputs $\mathbf{y}(\cdot,\cdot)$ take nonnegative values, A_1 and A_2 are nonnegative $n \times n$ matrices, B_1 and B_2 are nonnegative $n \times m$ matrices, while C is a nonnegative $p \times n$ matrix.

The initial conditions are assigned by specifying the (nonnegative) values of the state vectors $\mathbf{x}(h,k)$ on the initial *separation set* $\mathscr{S}_0 := \{(h,k) \in \mathbb{Z} \times \mathbb{Z} : h+k = 0\}$, namely by assigning all (the infinitely many) local states constituting the so-called initial *global state*

$$\mathscr{X}_0 := \{\mathbf{x}(h,k) : (h,k) \in \mathscr{S}_0\}.$$

While local properties easily lead to a “point by point” analysis in the discrete grid, and hence to characterizations [10] based on the (infinite family of) real matrices representing the way in which any input sample at (h,k) contributes to the state evolution at each point of the causality cone $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i \geqslant h, j \geqslant k\}$, their global counterparts naturally suggest a “separation set” viewpoint.

Moreover, the global state $\mathscr{X}_k := \{\mathbf{x}(h,k-h) : h \in \mathbb{Z}\}$ can be represented either by means of a 2D power series

$$X_k(z_1,z_2) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h,k-h) z_1^h z_2^{k-h}$$

or (once the information about the separation set $\mathscr{S}_k$ we are considering is known) by means of a 1D power series

$$X_k(\xi) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h, k-h)\xi^h,$$

and the state updating along the separation sets can be described, in turn, in 1D or 2D polynomial terms. So, global reachability and observability properties may be fruitfully addressed starting from a polynomial description.

With respect to standard 2D systems, the nonnegativity assumption rules out the possibility of resorting to those linear algebra tools commonly used when dealing with vector spaces. Indeed, all state, input and output variables lie in cones and this entails far deep consequences. On the one hand, the results available are typically weaker, as cones are not necessarily finitely generated and this occasionally prevents the existence of finite tests for structural properties. On the other hand, nonnegativity allows to resort to a combinatorial approach, as reachability and observability tests for 2D positive systems can be frequently translated into problems of path searching in suitable directed graphs.

The paper is organized as follows: Section 2 introduces some notations and provides both local and global reachability definitions. Local and global reachability are addressed in Sections 3 and 4, respectively. Finally, local and global observability are introduced and characterized in Section 5.

2. Preliminary concepts

Before proceeding, it is convenient to include some basic definitions and preliminary concepts that will be used in the paper. The Hurwitz products of two $n \times n$ matrices A_1 and A_2 are inductively defined [5] as

$$\begin{aligned}
&{A_1}^i \sqcup\!\sqcup^j A_2 = 0, && \text{if either } i \text{ or } j \text{ is negative},\\
&{A_1}^i \sqcup\!\sqcup^0 A_2 = A_1^i, && \text{if } i \geqslant 0,\\
&{A_1}^0 \sqcup\!\sqcup^j A_2 = A_2^j, && \text{if } j \geqslant 0,\\
&{A_1}^i \sqcup\!\sqcup^j A_2 = A_1({A_1}^{i-1} \sqcup\!\sqcup^j A_2) + A_2({A_1}^i \sqcup\!\sqcup^{j-1} A_2), && \text{if } i, j > 0.
\end{aligned}$$

Given a nonnegative vector $\mathbf{v} \in \mathbb{R}_+^n$, we define its *nonzero pattern* as the set $p(\mathbf{v}) := \{i \in \{1, 2, \ldots, n\} : v_i \neq 0\}$. Notice that the nonzero pattern of the sum of two nonnegative vectors in $\mathbb{R}_+^n$ is just the set union of their nonzero patterns, i.e., $p(\mathbf{v}_1 + \mathbf{v}_2) = p(\mathbf{v}_1) \cup p(\mathbf{v}_2)$. Similar definitions and remarks extend to nonnegative matrices. So, when the interest is just in the nonzero patterns of the nonnegative vectors/matrices, and not in the specific values of their nonzero entries, one can represent any vector $\mathbf{v} \in \mathbb{R}_+^n$ (any matrix $M \in \mathbb{R}_+^{p\times n}$) by means of the boolean vector $\mathbf{v}_B \in \{0, 1\}^n$ (the boolean matrix $M_B \in \{0, 1\}^{p\times n}$) having the same nonzero pattern as $\mathbf{v}$ (as M).

We denote by $\mathbf{1}_n$ the n-dimensional real vector with all entries equal to 1.

A polynomial vector $\mathbf{v}(z_1, z_2) \in \mathbb{R}_+[z_1, z_2]$ is said to be an ith *p-monomial vector* if $\mathbf{v}(1, 1)$ has the same nonzero pattern as the ith canonical vector $\mathbf{e}_i$, i.e., $p(\mathbf{v}(1, 1)) = \{i\}$, and its nonzero entry is a monomial $cz_1^h z_2^k$ in $\mathbb{R}_+[z_1, z_2]$. In the following, we will use the term ith p-monomial vector also for row vectors $\mathbf{v}^{\mathrm{T}}(z_1, z_2)$ such that $\mathbf{v}(z_1, z_2)$ is an ith p-monomial vector. A *p-monomial matrix* is a nonsingular (square) matrix whose columns are p-monomial vectors. P-monomial vectors and p-monomial matrices in $\mathbb{R}_+[\xi]$ are defined in an analogous way. Standard monomial vectors and monomial matrices in $\mathbb{R}_+$ can be seen as special cases of their general polynomial versions.

Local and global reachability properties are introduced in Definition 2.1, below.

Definition 2.1. A 2D state-space model (1.1)–(1.2) is

- (*positively*) *locally reachable* [5] if, upon assuming $\mathscr{X}_0 = 0$, for every $\mathbf{x}^* \in \mathbb{R}^n_+$ there exist $(h,k) \in \mathbb{Z} \times \mathbb{Z}$, with $h+k > 0$, and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ s.t. $\mathbf{x}(h,k) = \mathbf{x}^*$. When so, we will say that $\mathbf{x}^*$ is (locally) reachable in $h+k$ steps;
- (*positively*) *globally reachable* [5] if, upon assuming $\mathscr{X}_0 = 0$, for every global state $\mathscr{X}^*$ with entries in $\mathbb{R}^n_+$, there exist $k \in \mathbb{Z}_+$ and a nonnegative input sequence $\mathbf{u}(\cdot,\cdot)$ s.t. the global state $\mathscr{X}_k = \{\mathbf{x}(h,k-h) : h \in \mathbb{Z}\}$ coincides with $\mathscr{X}^*$. When so, we will say that $\mathscr{X}^*$ is (globally) reachable in k steps.

Notice that all nonnegative input sequences involved have supports included in the half-plane $\{(h,k) \in \mathbb{Z} \times \mathbb{Z} : h+k \geqslant 0\}$. In the following, the specification "positively" will be omitted when no ambiguities arise. Clearly, global reachability ensures local reachability.

3. Local reachability

When dealing with standard (i.e., not necessarily positive) 2D systems, local reachability is easily tested by evaluating the column span of the *reachability matrix in k steps* [5], i.e.

$$\begin{aligned}\mathscr{R}_k &= [B_1 \;\; B_2 \;\; A_1B_1 \;\; A_1B_2 + A_2B_1 \;\; A_2B_2 \;\; A_1^2B_1 \;\; (A_1{}^1 \sqcup\!\sqcup {}^1A_2)B_1 + A_1^2B_2 \;\; \cdots \;\; A_2^{k-1}B_2] \\ &= \left[(A_1{}^{i-1} \sqcup\!\sqcup {}^{j} A_2)B_1 + (A_1{}^{i} \sqcup\!\sqcup {}^{j-1} A_2)B_2\right]_{i,j \geqslant 0,\; 0<i+j \leqslant k}\end{aligned}$$

as k varies over the set of positive integers. Indeed, reachable states in k steps, i.e. local states that can be reached in any assigned position of the separation set $\mathscr{S}_k = \{(h,k-h) : h \in \mathbb{Z}\}$, starting from $\mathscr{X}_0 = 0$, constitute a linear subspace $\widetilde{X}_k \subseteq \mathbb{R}^n$, spanned by the columns of $\mathscr{R}_k$. Clearly, the ascending chain of reachability subspaces $\widetilde{X}_1 \subseteq \widetilde{X}_2 \subseteq \widetilde{X}_3 \subseteq \cdots \subseteq \mathbb{R}^n$ eventually reaches stationarity and this necessarily happens, by the 2D Cayley–Hamilton theorem (see e.g. [6]), in no more than n steps. As a consequence, if the 2D system is locally reachable, the point (h,k) where $\mathbf{x}(h,k)$ attains the desired value $\mathbf{x}^*$ (see Definition 2.1) can always be chosen on the separation set $\mathscr{S}_n$.

Once we constrain the system and the input sequence to be nonnegative, the reachability subspaces $\widetilde{X}_k$, $k \in \mathbb{N}$, are replaced by the *reachability cones* $\widetilde{X}_k^+$, $k \in \mathbb{N}$. In fact, the set $\widetilde{X}_k^+$ of all local states that can be reached in any assigned position of the separation set $\mathscr{S}_k$, by means of nonnegative inputs and starting from initial zero conditions ($\mathscr{X}_0 = 0$), obviously coincides with the set of all nonnegative combinations of the columns of $\mathscr{R}_k$, namely $\widetilde{X}_k^+ = \mathrm{Cone}(\mathscr{R}_k)$. Consequently, a system is locally reachable if and only if there exists $N \in \mathbb{N}$ s.t. $\mathrm{Cone}(\mathscr{R}_N) = \mathbb{R}^n_+$.

It is worth remarking that for a not locally reachable 2D positive system the chain of reachability cones does not necessarily reach stationarity and, indeed, certain positive states can be reached only asymptotically. However, as the nonzero patterns of the vectors in $\mathbb{R}^n_+$ constitute a finite set, there exists a positive integer k such that the nonzero patterns of all reachable local states are the set theoretic unions of the nonzero patterns of suitable columns of $\mathscr{R}_k$. Consequently, every nonzero pattern can be reached either in no more than k steps or never.

Positive local reachability is trivially equivalent to the possibility of reaching (starting from zero initial conditions) every vector of the canonical basis in $\mathbb{R}^n$ by means of nonnegative inputs, which in turn amounts to saying that there exists some $N \in \mathbb{N}$ s.t. the reachability matrix in N steps, $\mathscr{R}_N$, includes an $n \times n$ monomial submatrix.

Proposition 3.1 [10]. *Given a 2D system (1.1)–(1.2) the following facts are equivalent*:

(i) *the system is locally reachable*;
(ii) *there exist* n *pairs* $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n,$ *and* n *indices* $j = j(i) \in \{1, 2, \ldots, m\}$ *s.t.*
$$p\big((A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)B_1\mathbf{e}_j + (A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\mathbf{e}_j\big) = \{i\}; \tag{3.1}$$
(iii) *there exists* $N \in \mathbb{N}$ *such that the reachability matrix in* N *steps* $\mathscr{R}_N$ *has an* $n \times n$ *monomial submatrix*.

Remarks. (i) For 2D systems with scalar inputs, the aforementioned condition (3.1) simply becomes: there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n,$ s.t.
$$p\big((A_1{}^{h_i-1} \sqcup\!\sqcup^{k_i} A_2)B_1 + (A_1{}^{h_i} \sqcup\!\sqcup^{k_i-1} A_2)B_2\big) = \{i\}.$$
Notice, finally, that all pairs (h_i, k_i) are necessarily distinct, but the case may occur that $h_i + k_i = h_j + k_j$ for $i \neq j$.

(ii) Differently from the 1D case, we cannot ensure that the local reachability index N is bounded from above by the system dimension n [10].

4. Global reachability

When dealing with a polynomial description of the forced state evolution, the global state on the kth separation set
$$X_k(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{x}(h, k-h) z_1^h z_2^{k-h}$$
can be expressed in terms of the input sequences on the separation sets $\mathscr{S}_t$, $0 \leqslant t \leqslant k-1$, as follows:
$$X_k(z_1, z_2) = \mathscr{R}_k(z_1, z_2) \begin{bmatrix} U_{k-1}(z_1, z_2) \\ U_{k-2}(z_1, z_2) \\ \vdots \\ U_0(z_1, z_2) \end{bmatrix}, \tag{4.1}$$
where
$$\mathscr{R}_k(z_1, z_2) = \big[\, (B_1 z_1 + B_2 z_2) \quad (A_1 z_1 + A_2 z_2)(B_1 z_1 + B_2 z_2) \quad \cdots \quad (A_1 z_1 + A_2 z_2)^{k-1}(B_1 z_1 + B_2 z_2) \,\big] \tag{4.2}$$
and
$$U_t(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{u}(h, t-h) z_1^h z_2^{t-h}, \qquad t = 0, 1, \ldots, k-1.$$
Starting from this 2D polynomial description, we obtain the following characterization of global reachability.

Proposition 4.1. *The* 2*D system* (1.1)–(1.2) *is globally reachable if and only if there exists some nonnegative index N such that the* 2*D polynomial matrix* $\mathscr{R}_N(z_1, z_2)$ *given in* (4.2) *includes an* $n \times n$ *p-monomial submatrix*, *i.e.*

$$M \cdot \operatorname{diag}\{z_1^{\mu_1} z_2^{\nu_1}, z_1^{\mu_2} z_2^{\nu_2}, \ldots, z_1^{\mu_n} z_2^{\nu_n}\}, \tag{4.3}$$

for some monomial matrix M and some $\mu_i,\ \nu_i \geqslant 0,\ i = 1, 2, \ldots, n$.

Proof. It suffices to focus on the reachability of the "elementary global states" consisting of all zero (local) states except for one of them, which coincides with the monomial vector $\mathbf{e}_i$, $i \in \{1, 2, \ldots, n\}$. Indeed, if the system is globally reachable then, in particular, all elementary global states must be reachable. On the other hand, if all elementary global states are reachable, each of them can be reached by means of a suitable finite support nonnegative input sequence. So, by superposing nonnegative combinations of such finite support input sequences, one can reach every nonnegative global state. Consequently, the 2D system (1.1)–(1.2) is globally reachable if and only if, for every $i \in \{1, 2, \ldots, n\}$, there exists $k_i \in \mathbb{N}$ and $h_i \in \mathbb{Z}$ such that

$$X_{k_i}(z_1, z_2) = \mathbf{e}_i z_1^{h_i} z_2^{k_i - h_i}$$

is a global state reachable in k_i steps. This amounts to saying that

$$\mathbf{e}_i z_1^{h_i} z_2^{k_i - h_i} = \mathscr{R}_{k_i}(z_1, z_2) \begin{bmatrix} U_{k_i-1}^{(i)}(z_1, z_2) \\ U_{k_i-2}^{(i)}(z_1, z_2) \\ \vdots \\ U_0^{(i)}(z_1, z_2) \end{bmatrix},$$

for some $U_t^{(i)}(z_1, z_2)$, $t = 0, 1, \ldots, k_i - 1$. It entails no loss of generality assuming that each $U_t^{(i)}(z_1, z_2)$ has finite support, namely it is a Laurent polynomial. The nonnegativity of the coefficients of all polynomial matrices and vectors involved ensures that the above condition holds true if and only if there exists at least one column of $\mathscr{R}_{k_i}(z_1, z_2)$ taking the following structure $\mathbf{e}_i c_i z_1^{\mu_i} z_2^{\nu_i}$, $c_i \in \mathbb{R}_+$. So, the proposition statement holds for $N = \max\{k_1, k_2, \ldots, k_n\}$. □

Remark. The characterization given in Proposition 4.1 above, may be restated in terms of polynomial reachability matrices in the single variable ξ. Indeed, the 2D system (1.1)–(1.2) is globally reachable if and only if there exists $N \in \mathbb{N}$ such that

$$\mathscr{R}_N(\xi) = [\,(B_1 + B_2\xi) \quad (A_1 + A_2\xi)(B_1 + B_2\xi) \quad \ldots \quad (A_1 + A_2\xi)^{N-1}(B_1 + B_2\xi)\,]$$

includes an $n \times n$ p-monomial submatrix, i.e.:

$$M \cdot \operatorname{diag}\{\xi^{\nu_1}, \xi^{\nu_2}, \ldots, \xi^{\nu_n}\},$$

for some monomial matrix M and some $\nu_i \geqslant 0$, $i = 1, 2, \ldots, n$.

As an immediate corollary of the previous result, we get

Corollary 4.2. *If the* 2*D system* (1.1)–(1.2) *is globally reachable then*

$$[\,A_1 + A_2\xi \quad B_1 + B_2\xi\,]$$

includes an $n \times n$ *p-monomial matrix*.

Proof. The nonnegativity of the coefficients, together with the fact that the product of two polynomials is a monomial if and only if they are both monomials, allows to saying that

$$(A_1 + A_2\xi)^{k_i-1}(B_1 + B_2\xi)\mathbf{e}_j = \mathbf{e}_i \cdot (c_i \cdot \xi^{\mu_i}) \tag{4.4}$$

$c_i \in \mathbb{R}_+$, $\mu_i \in \mathbb{Z}_+$, implies either that the *j*th column of $B_1 + B_2\xi$ is an *i*th p-monomial vector (if $k_i = 1$) or that some column of $A_1 + A_2\xi$ is an *i*th p-monomial vector (if $k_i > 1$). Since (4.4) must be verified for every $i \in \{1, 2, \ldots, n\}$, the result immediately follows. □

The following lemma leads the way to further characterizations of global reachability.

Lemma 4.3. *If the* 2*D system* (1.1)–(1.2) *is globally reachable then the* 1*D positive system described by the pair* $(A_1 + A_2, B_1 + B_2)$ *is* (*positively*) *reachable.*

Proof. By the previous Proposition 4.1, the 2D system (1.1)–(1.2) is globally reachable if and only if there exists some positive index *N* such that the 2D polynomial matrix $\mathscr{R}_N(\xi)$ includes an $n \times n$ p-monomial submatrix, i.e.:

$$M \cdot \text{diag}\{\xi^{\nu_1}, \xi^{\nu_2}, \ldots, \xi^{\nu_n}\},$$

for some monomial matrix *M* and some $\nu_i \geqslant 0$, $i = 1, 2, \ldots, n$. Since this condition holds true for an arbitrary ξ, then it must hold true for $\xi = 1$. This means that the reachability matrix in *N* steps of the pair $(A_1 + A_2, B_1 + B_2)$ includes the monomial submatrix *M* and hence [3] the pair $(A_1 + A_2, B_1 + B_2)$ is positively reachable. □

Lemma 4.4. *If there exist an integer* $\ell \in \mathbb{Z}_+$ *and a nonzero polynomial* $p(\xi) \in \mathbb{R}_+[\xi]$ *such that*

$$(A_1 + A_2\xi)^{\ell}(B_1 + B_2\xi)\mathbf{e}_j = p(\xi)\mathbf{e}_i$$

for some indices $i \in \{1, 2, \ldots, n\}$ *and* $j \in \{1, 2, \ldots, m\}$, *then there exists* $\bar{\ell} \in \mathbb{Z}_+$, $0 \leqslant \bar{\ell} \leqslant n - 1$, *and a nonzero* $\bar{p}(\xi) \in \mathbb{R}_+[\xi]$ *such that*

$$(A_1 + A_2\xi)^{\bar{\ell}}(B_1 + B_2\xi)\mathbf{e}_j = \bar{p}(\xi)\mathbf{e}_i.$$

Proof. We associate with the polynomial matrix pair $(A_1 + A_2\xi, B_1 + B_2\xi)$ a directed graph with *n* vertices and *m* sources: there is an arc connecting vertex *j* to vertex *i* if and only if the (i, j)th entry of $A_1 + A_2\xi$ is nonzero and, similarly, there is an arc connecting source *j* to vertex *i* if and only if the (i, j)th entry of $B_1 + B_2\xi$ is nonzero. Each arc is thus weighted by some nonzero polynomial $c + d\xi$, $c, d \in \mathbb{R}_+$.

In graph theoretic terms, condition

$$(A_1 + A_2\xi)^{\ell}(B_1 + B_2\xi)\mathbf{e}_j = p(\xi)\mathbf{e}_i$$

holds for some nonzero polynomial $p(\xi) \in \mathbb{R}_+[\xi]$ if and only if there exists a "deterministic path" [10,15] from the *j*th source to the *i*th vertex in the directed graph associated with the pair $(A_1 + A_2\xi, B_1 + B_2\xi)$, namely a path starting from the *j*th source and reaching, after $\ell + 1$ steps, the vertex *i* and no other vertex. Such a condition holds, in turn, if and only if there exists a deterministic path from the *j*th source to the *i*th vertex in the directed graph associated with $(A_1 + A_2, B_1 + B_2)$. But then, we may resort to the result obtained by Coxson and Larson in [2] and say that if such a path exists, then there exists a path from source *j* to vertex *i* of length not larger than *n*. This corresponds to saying that

$$(A_1 + A_2)^{\bar{\ell}}(B_1 + B_2)\mathbf{e}_j = c \cdot \mathbf{e}_i$$

holds true for some $\bar{\ell} < n$ and some $c > 0$ and hence that

$$(A_1 + A_2\xi)^{\bar{\ell}}(B_1 + B_2\xi)\mathbf{e}_j = \bar{p}(\xi)\mathbf{e}_i$$

holds true for some $\bar{\ell} < n$ and some nonzero polynomial $\bar{p}(\xi) \in \mathbb{R}_+[\xi]$. $\square$

Proposition 4.5. *The 2D system (1.1)–(1.2) of size n is globally reachable if and only if the 2D polynomial matrix $\mathscr{R}_n(\xi)$ includes an $n \times n$ p-monomial submatrix.*

Proof. By Proposition 4.1, the system is globally reachable if and only if there exists some $N \in \mathbb{N}$ such that the 2D polynomial matrix $\mathscr{R}_N(\xi)$ includes an $n \times n$ p-monomial submatrix. However, by the previous Lemma 4.4, if

$$(A_1 + A_2\xi)^{\ell}(B_1 + B_2\xi)\mathbf{e}_j = c \cdot \xi^{\nu_i}\mathbf{e}_i$$

for some nonzero monomial $c \cdot \xi^{\nu_i} \in \mathbb{R}_+[\xi]$, then there exists $\bar{\ell} \in \mathbb{Z}_+$, $0 \leqslant \bar{\ell} \leqslant n-1$ and a nonzero $\bar{p}(\xi) \in \mathbb{R}_+[\xi]$ such that

$$(A_1 + A_2\xi)^{\bar{\ell}}(B_1 + B_2\xi)\mathbf{e}_j = \bar{p}(\xi)\mathbf{e}_i.$$

This implies, in particular, that

$$(A_1 + A_2\xi)^{\ell-\bar{\ell}}[(A_1 + A_2\xi)^{\bar{\ell}}(B_1 + B_2\xi)\mathbf{e}_j] = (A_1 + A_2\xi)^{\ell-\bar{\ell}}[\bar{p}(\xi)\mathbf{e}_i] = c \cdot \xi^{\nu_i}\mathbf{e}_i.$$

As a consequence, the *i*th column of $(A_1 + A_2\xi)^{\ell-\bar{\ell}}$ must be an *i*th p-monomial vector, $\bar{p}(\xi)$ is necessarily a monomial, and there exists an *i*th monomial column $\bar{p}(\xi)\mathbf{e}_i$ in $\mathscr{R}_n(\xi)$. $\square$

Remarks. (i) For 2D systems with scalar inputs, the proof of Proposition 4.5 is much easier, as indeed Lemma 4.4 is unnecessary. Global reachability of a 2D system with scalar inputs ensures, by Lemma 4.3, that the reachability matrix in *n* steps associated with the pair $(A_1 + A_2, B_1 + B_2)$ is monomial and hence

$$\mathscr{R}_n(\xi) = M \cdot \operatorname{diag}\{p_1(\xi), p_2(\xi), \ldots, p_n(\xi)\}$$

for some monomial matrix *M* and some polynomials $p_i(\xi) \in \mathbb{R}_+[\xi]$, $i = 1, 2, \ldots, n$. But then, one can follow the same reasoning adopted within the previous proof and say that

$$(A_1 + A_2\xi)^{\ell}(B_1 + B_2\xi) = c \cdot \xi^{\nu_i}\mathbf{e}_i$$

for some $\ell \geqslant n$ implies that the column of $\mathscr{R}_n(\xi)$ having a nonzero entry only in the *i*th position is already an *i*th p-monomial vector.

(ii) The nonzero pattern of any global state $\mathscr{X}_k$ on the separation set $\mathscr{S}_k$ is defined as

$$\mathscr{P}(\mathscr{X}_k) := \{(i, j) \in \{1, 2, \ldots, n\} \times \mathbb{Z} : x_i(j, k-j) \neq 0\}.$$

Since all global states of a globally reachable 2D system can be obtained on the separation set $\mathscr{S}_n$, every subset of $\{1, 2, \ldots, n\} \times \mathbb{Z}$ represents the nonzero pattern of a global state that can be reached on $\mathscr{S}_n$. On the other hand, if a 2D positive system is not globally reachable, the families of the nonzero patterns of the reachable global states on $\mathscr{S}_k$ may constitute a chain that strictly increases when *k* goes to $+\infty$, as shown by the following example.

Example. Consider the 2D positive system described by the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since nonzero patterns of the global states are independent of the specific nonzero values taken by the matrix entries and by the inputs sequences, for the sake of simplicity we will represent nonzero entries as units or, equivalently, we will assume that all entries belong to the boolean algebra {0, 1}. Consequently, the reachability matrix in $k+2$ steps is given by

$$\mathscr{R}_{k+2}(\xi) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & \xi & \xi^2 & \xi^3 & \cdots & \xi^{k+1} \\ 0 & 0 & \xi & \xi+\xi^2 & \cdots & \xi+\xi^2+\cdots+\xi^k \end{bmatrix}.$$

It is clear that for every $k \in \mathbb{N}$ the global state

$$\mathbf{e}_2 + \mathbf{e}_4(\xi+\xi^2+\cdots+\xi^k) + \mathbf{e}_3\xi^{k+1}$$

can be reached only after $k+2$ steps, and therefore, as k increases, the nonzero patterns of the reachable global states constitute a strictly ascending chain of subsets in $\{1, 2, 3, 4\} \times \mathbb{Z}$.

A nice polynomial canonical form can be obtained for globally reachable systems with scalar inputs.

Proposition 4.6. *For a* $2D$ *system* (1.1)–(1.2) *of size n with scalar inputs the following facts are equivalent*:

(i) *the system is globally reachable*;
(ii) *there exists a permutation matrix P such that*

$$P^{\mathrm{T}}(A_1 + A_2\xi)P = \begin{bmatrix} \star & a_{12} & & & 0 \\ \star & 0 & a_{23} & & 0 \\ \star & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ \star & 0 & \dots & 0 & 0 \end{bmatrix}, \quad P^{\mathrm{T}}(B_1 + B_2\xi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix}, \tag{4.5}$$

where $a_{i,i+1}, b_n \in \mathbb{R}_+[\xi]$ *are nonzero monomials and* $\star$ *denotes a polynomial in* $\mathbb{R}_+[\xi]$ *of degree at most* 1;
(iii) *the* $2D$ *polynomial matrix* $\mathscr{R}_n(\xi)$ *is a p-monomial matrix*;
(iv) $\mathscr{R}_n(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ *is nonsingular and* $\mathscr{R}_n(\xi)^{-1}$ *belongs to* $\mathbb{R}_+[\xi^{-1}]^{n\times n}$.

Proof. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are obvious.

(i) $\Rightarrow$ (ii) By Corollary 4.2, global reachability ensures that the $n \times (n+1)$ polynomial matrix

$$[\, A_1 + A_2\xi \quad B_1 + B_2\xi \,]$$

includes an $n \times n$ p-monomial matrix. Suppose that $B_1 + B_2\xi$ is not a monomial vector. Since it cannot be zero, then it must either have at least two nonzero entries (case 1) or be a vector of the following type $p(\xi)\mathbf{e}_i$, for some polynomial $p(\xi)$ of lag 1 (case 2). On the other hand, since the block matrix must include an $n \times n$ p-monomial matrix, such a matrix must be $A_1 + A_2\xi$.

It is easily seen that, under these hypotheses, both in case 1 and in case 2, none of the vectors $(A_1 + A_2\xi)^k(B_1 + B_2\xi)$ can be p-monomial. So, $B_1 + B_2\xi$ is necessarily a p-monomial vector. It entails no loss of generality assuming $B_1 + B_2\xi = \mathbf{e}_n$. In fact, we can always reduce ourselves to this case by permuting either the vector components or the matrices B_1 and B_2, possibly both. Clearly, at most one column of $A_1 + A_2\xi$ is not p-monomial and the set of the remaining $n-1$ columns of $A_1 + A_2\xi$ includes an ith p-monomial vector for $i = 1, 2, \ldots, n-1$. Suppose that the last column of $A_1 + A_2\xi$ is not p-monomial. This implies that, on the one hand, all remaining columns of $A_1 + A_2\xi$ are p-monomial, on the other hand $(A_1 + A_2\xi)(B_1 + B_2\xi)$ is not p-monomial. For these reasons, both in case $(A_1 + A_2\xi)(B_1 + B_2\xi)$ has at least two nonzero entries (case 1) and in case it is a vector like $p(\xi)\mathbf{e}_i$, for some polynomial $p(\xi)$ of lag 1 (case 2), also the following powers $(A_1 + A_2\xi)^i(B_1 + B_2\xi)$, $i > 1$, are not p-monomial. Suppose now that $(A_1 + A_2\xi)(B_1 + B_2\xi)$ is a p-monomial vector. Clearly the nonzero entry cannot be in the last (namely nth) position, otherwise all powers $(A_1 + A_2\xi)^i(B_1 + B_2\xi)$ would have the same structure, and it entails no loss of generality assuming that the only nonzero entry lies in the $n-1$th row. We can now repeat the same reasoning we just applied to the last row and claim that if the $n-1$th column would not be p-monomial then all the other columns in $A_1 + A_2\xi$ would not be, and hence all remaining powers $(A_1 + A_2\xi)^i(B_1 + B_2\xi)$, $i \geqslant 2$, would not be p-monomial. In this way we have proven that (upon a suitable permutation) we can assume that all columns of $A_1 + A_2\xi$, except possibly for the first one, have to be p-monomial vectors and

$$A_1 + A_2\xi = \begin{bmatrix} \bigstar & a_{12} & & & 0 \\ \bigstar & 0 & a_{23} & & 0 \\ \bigstar & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ \bigstar & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad B_1 + B_2\xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix}, \tag{4.6}$$

where $a_{i,i+1}, b_n \in \mathbb{R}_+[\xi]$ are nonzero monomials and the entries denoted by $\bigstar$ are polynomials in $\mathbb{R}_+[\xi]$ (of degree at most 1).

(iii) $\Leftrightarrow$ (iv) Suppose that $\mathscr{R}_n(\xi)$ belongs to $\mathbb{R}_+[\xi]^{n\times n}$ and its inverse $\mathscr{R}_n(\xi)^{-1}$ to $\mathbb{R}_+[\xi^{-1}]^{n\times n}$. So, at each point $\bar{\xi} \in \mathbb{R}_+$ $\mathscr{R}_n(\bar{\xi})$ and $\mathscr{R}_n(\bar{\xi})^{-1}$ are nonnegative matrices satisfying $I_n = \mathscr{R}_n(\bar{\xi})$ $\mathscr{R}_n(\bar{\xi})^{-1}$. Since the only nonnegative square matrices endowed with nonnegative inverses are monomial, this implies that $\mathscr{R}_n(\bar{\xi})$ is monomial for every $\bar{\xi} \in \mathbb{R}_+$. This is possible (if and) only if $\mathscr{R}_n(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ is p-monomial and hence satisfies (iii). The converse is obvious. $\square$

5. Local and global observability

Global and local reachability definitions have been introduced in Section 2 by referring to arbitrary global or local nonnegative states to be reached (starting from zero global initial conditions) by means of nonnegative inputs. It can be shown that referring to the nonzero patterns of the (global/local) states and of the input sequences (instead of considering their specific nonnegative values) leads to reachability definitions that are exactly equivalent to those given in Section 2 and, consequently, a nonzero pattern approach to positive reachability is just an alternative way for introducing the same concepts.

Definition 5.1. A 2D state-space model (1.1)–(1.2) is

- *locally reachable* if, upon assuming $\mathscr{X}_0 = 0$, for every boolean vector $\mathbf{x}_B^* \in \{0, 1\}^n$ there exist $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, with $h + k > 0$, and a boolean input sequence $\mathbf{u}(\cdot, \cdot)$ s.t. $\mathbf{x}(h, k)$ has the same nonzero pattern as $\mathbf{x}_B^*$;
- *globally reachable* if, upon assuming $\mathscr{X}_0 = 0$, for every sequence of n-dimensional boolean vectors $\{\mathbf{x}_B(h)\}_{h\in\mathbb{Z}}$, there exist $k \in \mathbb{Z}_+$ and a boolean input sequence $\mathbf{u}(\cdot, \cdot)$ determining a global state $\mathscr{X}_k = \{x(h, k - h), h \in \mathbb{Z}\}$ on $\mathscr{S}_k$ with
$$\mathscr{P}(\mathscr{X}_k) = \mathscr{P}(\{\mathbf{x}_B(h)\}_{h\in\mathbb{Z}}),$$
namely the nonzero patterns of the two sequences ordinately coincide.

On the other hand, if we aim at introducing observability definitions starting from the free output evolutions of 2D positive systems, and pretend that they provide reasonable dual properties w.r.t local and global reachability, a nonzero pattern approach is somehow unavoidable.

Definition 5.2. A 2D state-space model (1.1)–(1.2) is

- *locally observable* if, upon assuming that the initial global state $\mathscr{X}_0$ consists of a single nonzero local state $\mathbf{x}(0, 0)$, the knowledge of the nonzero pattern of the free output evolution $\mathbf{y}_\ell(h, k)$ in every point $(h, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ allows to uniquely determine the nonzero pattern of $\mathbf{x}(0, 0)$;
- *globally observable* if the knowledge of the nonzero pattern of the free output evolution $\mathbf{y}_\ell(h, k)$ in every point $(h, k) \in \mathbb{Z} \times \mathbb{Z}, h + k \geqslant 0$, allows to uniquely determine the nonzero pattern of the initial global state $\mathscr{X}_0$.

It is easy to see that global observability trivially implies local observability, as, indeed, among all possible initial global states one may consider those consisting of all zero local states except at $(0, 0)$, and, corresponding to that type of global states, the support of the free output evoluation is included in the first orthant.

In order to explore local observability, we introduce the *observability matrix in k steps*, i.e.

$$\mathscr{O}_k = \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ C(A_1{}^1 \sqcup\!\sqcup^1 A_2) \\ CA_2^2 \\ \vdots \\ CA_2^{k-1} \end{bmatrix} = \Big[C(A_1{}^i \sqcup\!\sqcup^j A_2)\Big]_{i,j\geqslant 0,\ 0\leqslant i+j<k},$$

where k is a positive integer. As a first step, we provide a characterization of local observability.

Proposition 5.3. *Given a* 2*D system* (1.1)–(1.2) *the following facts are equivalent*:

(i) *the system is locally observable*;
(ii) *there exist n pairs* $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+, i = 1, 2, \ldots, n$, *and n indices* $j = j(i) \in \{1, 2, \ldots, p\}$ *s.t.*

$$p(\mathbf{e}_j^{\mathrm{T}} C(A_1{}^{h_i} \sqcup\!\sqcup^{k_i} A_2)) = \{i\}; \tag{5.1}$$

(iii) *there exists $N \in \mathbb{N}$ s.t. the observability matrix in N steps $\mathcal{O}_N$ has an $n \times n$ monomial submatrix.*

Proof. (i) $\Rightarrow$ (ii) Suppose, by contradiction, that the system is locally observable but ii) does not hold. This means that there exists $\ell \in \{1, 2, \ldots, n\}$ s.t. none of the rows of the observability matrix in k steps, for any $k \in \mathbb{N}$, is an ℓth monomial vector. It is easy to verify that the initial states $\mathbf{x}(0, 0) = \mathbf{1}_n$ and $\mathbf{x}(0, 0) = \mathbf{1}_n - \mathbf{e}_\ell$ have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns. Thus the system cannot be locally observable.

(ii) $\Rightarrow$ (i) If (ii) holds true, the ith entry of the local state $\mathbf{x}(0, 0)$ is nonzero if and only if $\mathbf{e}_j^{\mathrm{T}} \mathbf{y}_\ell(h_i, k_i) \neq 0$, $i = 1, 2, \ldots, n$. So, the system is locally observable.

The equivalence of (ii) and (iii) is obvious. $\square$

When dealing with 2D systems with scalar outputs, condition (ii) above simply becomes: there exist n pairs $(h_i, k_i) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $i = 1, 2, \ldots, n$, s.t. $C(A_1{}^{h_i} \sqcup\!\sqcup^{k_i} A_2)$ is an ith (row) monomial vector. Notice that, also in this case, all pairs (h_i, k_i) are necessarily distinct, but the case may occur that $h_i + k_i = h_j + k_j$ for $i \neq j$.

In order to address global observability by means of polynomial techniques, we express the free output evolution on each separation set $\mathscr{S}_t$ by means of a power series:

$$Y_t(z_1, z_2) = \sum_{h \in \mathbb{Z}} \mathbf{y}(h, t-h) z_1^h z_2^{t-h}$$

and relate it to the global initial conditions on the separation set $\mathscr{S}_0$ as follows:

$$\begin{aligned} Y_t(z_1, z_2) &= \sum_{h \in \mathbb{Z}} C\mathbf{x}(h, t-h) z_1^h z_2^{t-h} \\ &= \sum_{h \in \mathbb{Z}} C \sum_{\ell=0}^{t} (A_1{}^{\ell} \sqcup\!\sqcup^{t-\ell} A_2)\mathbf{x}(h-\ell, \ell-h) z_1^h z_2^{t-h} \\ &= C \sum_{\ell=0}^{t} (A_1{}^{\ell} \sqcup\!\sqcup^{t-\ell} A_2) \left(\sum_{h \in \mathbb{Z}} \mathbf{x}(h-\ell, \ell-h) z_1^{h-\ell} z_2^{\ell-h} \right) z_1^{\ell} z_2^{t-\ell} \\ &= C \sum_{\ell=0}^{t} (A_1{}^{\ell} \sqcup\!\sqcup^{t-\ell} A_2) z_1^{\ell} z_2^{t-\ell} X_0(z_1, z_2) \\ &= C(A_1 z_1 + A_2 z_2)^t X_0(z_1, z_2). \end{aligned}$$

Consequently

$$\begin{bmatrix} Y_0(z_1, z_2) \\ \vdots \\ Y_{k-1}(z_1, z_2) \end{bmatrix} = \mathcal{O}_k(z_1, z_2) X_0(z_1, z_2), \tag{5.2}$$

where

$$\mathcal{O}_k(z_1, z_2) := \begin{bmatrix} C \\ C(A_1 z_1 + A_2 z_2) \\ \vdots \\ C(A_1 z_1 + A_2 z_2)^{k-1} \end{bmatrix}. \tag{5.3}$$

Starting from this 2D polynomial description, we can obtain a characterization of global observability.

Proposition 5.4. *The* $2D$ *system* (1.1)–(1.2) *is globally observable if and only if there exists some nonnegative index* N *such that the* $2D$ *polynomial matrix* $\mathcal{O}_N(z_1, z_2)$ *given in* (5.3) *includes an* $n \times n$ *p-monomial submatrix.*

Proof. Of course, if there exists some index $N \in \mathbb{N}$ such that the observability matrix $\mathcal{O}_N(z_1, z_2)$ includes an $n \times n$ p-monomial submatrix, then there exists a suitable selection of separation sets $\mathcal{S}_{k_1}, \mathcal{S}_{k_2}, \dots, \mathcal{S}_{k_n}$, $k_i \in \mathbb{N}$ and a corresponding suitable choice of output components $j_1, j_2, \dots, j_n \in \{1, 2, \dots, p\}$ such that

$$\begin{bmatrix} \mathbf{e}_{j_1}^{\mathrm{T}} Y_{k_1}(z_1, z_2) \\ \mathbf{e}_{j_2}^{\mathrm{T}} Y_{k_2}(z_1, z_2) \\ \vdots \\ \mathbf{e}_{j_n}^{\mathrm{T}} Y_{k_n}(z_1, z_2) \end{bmatrix} = M \cdot \mathrm{diag}\{z_1^{\mu_1} z_2^{\nu_1}, z_1^{\mu_2} z_2^{\nu_2}, \dots, z_1^{\mu_n} z_2^{\nu_n}\} X_0(z_1, z_2).$$

Since we have already seen that a p-monomial matrix in $\mathbb{R}_+[\xi]^{n\times n}$ exhibits an inverse (in $\mathbb{R}_+[\xi^{-1}]^{n\times n}$) having the same structure, it follows that

$$\mathrm{diag}\{z_1^{-\mu_1} z_2^{-\nu_1}, z_1^{-\mu_2} z_2^{-\nu_2}, \dots, z_1^{-\mu_n} z_2^{-\nu_n}\} M^{-1} \cdot \begin{bmatrix} \mathbf{e}_{j_1}^{\mathrm{T}} Y_{k_1}(z_1, z_2) \\ \mathbf{e}_{j_2}^{\mathrm{T}} Y_{k_2}(z_1, z_2) \\ \vdots \\ \mathbf{e}_{j_n}^{\mathrm{T}} Y_{k_n}(z_1, z_2) \end{bmatrix} = X_0(z_1, z_2).$$

This allows an entry by entry identification of all local components of the initial global state, and hence the identification of the nonzero pattern of $\mathcal{X}_0$.

Conversely, suppose by contradiction that the system is globally observable but there exists some index $\ell \in \{1, 2, \dots, n\}$ such that none of the rows of the observability matrix in k steps $\mathcal{O}_k(z_1, z_2)$, for any $k \in \mathbb{N}$, is an ℓth p-monomial vector. Two cases may occur: either every row having a nonzero ℓth entry has also other nonzero entries, or all rows whose only nonzero entry is the ℓth one, exhibit a polynomial of strictly positive lag in the ℓth position. If so, we denote by $L > 0$ the smallest such lag. In the first case, it is easy to see that the initial global states

$$X_0(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{1}_n z_1^h z_2^{-h},$$
$$\bar{X}_0(z_1, z_2) = \sum_{h\in\mathbb{Z}} (\mathbf{1}_n - \mathbf{e}_\ell)\, z_1^h z_2^{-h}$$

have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns, thus contradicting global observability. Similarly, in the second case, the initial global states

$$X_0(z_1, z_2) = \sum_{h\in\mathbb{Z}} \mathbf{1}_n z_1^h z_2^{-h},$$

$$\bar{X}_0(z_1, z_2) = \sum_{h\in\mathbb{Z}} (\mathbf{1}_n - \mathbf{e}_\ell) z_1^h z_2^{-h} + \sum_{h\in\mathbb{Z}} \mathbf{e}_\ell z_1^{h(L+1)} z_2^{-h(L+1)}$$

have different nonzero patterns but produce free output evolutions endowed with the same nonzero patterns, contradicting again global observability. □

Remark. As for global reachability, the equivalent characterization given in Proposition 5.4 above, may be restated in terms of polynomial reachability matrices in the single variable ξ. Indeed, the 2D system (1.1)–(1.2) is globally observable if and only if there exists $N \in \mathbb{N}$ such that

$$\mathscr{O}_N(\xi) = \begin{bmatrix} C \\ C(A_1 + A_2\xi) \\ \vdots \\ C(A_1 + A_2\xi)^{N-1} \end{bmatrix}$$

includes an $n \times n$ p-monomial submatrix.

Also, as an immediate corollary of the previous result, we get.

Corollary 5.5. *If the* 2*D system* (1.1)–(1.2) *is globally observable then*

$$\begin{bmatrix} A_1 + A_2\xi \\ C \end{bmatrix}$$

includes an $n \times n$ p-monomial matrix.

Starting from Proposition 5.4, it is straightforward to apply the same type of reasonings adopted in Section 4 for global reachability, thus obtaining

Proposition 5.6. *The* 2*D system* (1.1)–(1.2) *of size n is globally observable if and only if the* 2*D polynomial matrix $\mathscr{O}_n(\xi)$ includes an $n \times n$ p-monomial submatrix.*

A nice polynomial canonical form can also be obtained for globally observable systems with scalar outputs, by resorting to the results derived in this section and to the reasonings adopted within the proof of Proposition 4.6.

Proposition 5.7. *For a* 2*D system* (1.1)–(1.2) *of size n with scalar outputs the following facts are equivalent*:

(i) *the system is globally observable*;
(ii) *there exists a permutation matrix P such that*

$$P^{\mathrm{T}}(A_1 + A_2\xi)P = \begin{bmatrix} \star & \star & \star & \dots & \star \\ a_{21} & 0 & 0 & & 0 \\ 0 & a_{32} & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \dots & a_{n,n-1} & 0 \end{bmatrix},$$
$$CP = [\,0 \quad 0 \quad 0 \quad \dots \quad c_n\,],$$

where $a_{i,i-1} \in \mathbb{R}_+[\xi]$ *are nonzero monomials*, $c_n \in \mathbb{R}_+$, $c_n > 0$, *and* ★ *denotes a polynomial in* $\mathbb{R}_+[\xi]$ *of degree at most* 1;

(iii) *the* $2D$ *polynomial matrix* $\mathcal{O}_n(\xi)$ *is a p-monomial submatrix*;

(iv) $\mathcal{O}_n(\xi) \in \mathbb{R}_+[\xi]^{n\times n}$ *is nonsingular and its inverse* $\mathcal{O}_n(\xi)^{-1}$ *belongs to* $\mathbb{R}_+[\xi^{-1}]^{n\times n}$.

It is worthwhile, at this point, to briefly comment on the duality relation existing between (global/local) reachability and (global/local) observability. Clearly 2D systems lack the completely symmetric structure 1D systems are endowed with, as they have two input-to-state matrices and a unique state-to-output matrix. However, if we assume (as in certain examples of [10]) $B_1 = B$ and $B_2 = 0$, then all previous relations are clearly dual. In the general case, duality holds in a weaker sense, by this meaning that there is an obvious correspondence between the various characterizations, which can be obtained one from the other provided that we suitably replace the pair (B_1, B_2) with the matrix C and suitably adjust the indices of the Hurwitz products.

6. Conclusions

In the paper, reachability and observability for 2D positive systems are investigated in their local and global versions. Significantly enough, their characterizations exhibit strict relationships both with the reachability and observability characterizations for standard (i.e. not necessarily positive) 2D systems and with those available for 1D positive systems. Indeed, as in the standard 2D case, local reachability and observability naturally involve real matrices in their description, while the corresponding global properties naturally rely on suitable polynomial matrices [5]. Moreover, as in the 1D case, all properties turn out to be related to the existence of a p-monomial submatrix within the reachability and observability matrices [3,13].

The necessary and sufficient conditions obtained for the 4 properties take just the same form, but pertain different (reachability/observability and real/polynomial) matrices, thus providing a nice general framework where further investigations can be developed. The only significant difference to remark is that, while global properties can be tested basing on certain block matrices whose block number does not exceed the 2D system dimension, local properties may require to evaluate quite large block matrices, and no upper bound on the block number is available up to now.

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